You Think You've Got Problems?

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Introduction. My life, like that of any other's, has its share of problems. I consider myself fortunate, however, to have more problems than most. In the past several years I've dealt with war, gambling, and drugs to name but a few. In writing this article I intend to dump some of my problems on you. The problems that I refer to are mathematical, but accessible to a wider audience (so keep reading). Several problems will be surveyed, with an overview of their origins and a brief discussion of the processes involved in their solutions. Computations will be kept to a minimum and there will be plenty of visuals to aid you in understanding the major ideas. Some of these problems have led to publications, others to presentations at conferences; some are still works in progress. Hopefully, you will find these problems interesting in their own right. At least, this article will shed a little light on what kind of problems some mathematics professors experience outside the classroom.

My interest in problem solving (as opposed to the hard-core theoretical research involved in a Ph.D.) began soon after I began working with mathematics majors. In mathematics it is very difficult, if not impossible, to engage undergraduate students in the traditional research that a faculty member might be involved in. A typical undergraduate will not have the background to understand the questions asked, let alone get excited about trying to answer them. Fortunately, there is no shortage of interesting problems that arise from everyday experience—problems that students can easily relate to and be intrigued by. These problems are "elementary" in the sense that they are easy to state, and require no mathematics more advanced than a student would ordinarily see in the course of the major (and often quite a bit less). On the other hand, they are good problems. Their solutions are not known, they are open-ended, can be difficult to solve and they often lead to other avenues of investigation. Teaching at liberal arts colleges, such problems have become central to my work. They have given me joy and struggles, led to publications and undergraduate research projects, and allowed me to share my love of mathematics with students and colleagues alike.

War. My ventures into problem solving began in earnest about 14 years ago, when playing the card game "War" with one of my sons. Adam, who was 4 years old at the time, was somewhat impatient while waiting for the "battles" (when the two players' cards have the same face value), and would make declarations such as "I'm not going to bed until after another battle." I, on the other hand, would be thinking more along the lines of "isn't this game ever going to end?" Such innocent thoughts and comments can be the seeds of serious work to someone in mathematics. Consider that battles in the game of War are a generalization of the well-known Hat-Check Problem (see [**3**], for example):

> Suppose ten gentlemen check their hats upon arrival at a restaurant. At the end of the meal, the hats are returned to them in a random order. What is the probability that none of the men get the correct hat?

If we restate the Hat-Check Problem in terms of cards, we get a simplified version of War:

> Suppose you have ten cards in your hand, the ace through ten of spades. Your opponent holds the ace through ten of hearts. You each shuffle your cards then lay them down one at a time. What is the probability of no matches (battles)?

The solution is $\frac{16481}{44800} \approx 0.367879464286$, so that there is a greater than 1 in 3 chance that no one gets the correct hat. Furthermore, as the number of gentlemen checking hats increases without bound, the solutions approach $\frac{1}{e} \approx 0.367879441171$. (As an aside, the fact that these two numbers are so close to each other indicates that for all practical purposes the answers are the same whether ten hats or ten thousand hats are checked!)

Studying battles in the game of War is much more complicated for two reasons. First, there are four suits instead of two, allowing for multiple possible matches (imagine ten *couples* checking hats and considering whether or not each person is returned a hat that belongs to them or their partner). Second, in a typical deal of the cards one player might hold all the kings, for example, so that no match of kings is possible. It turns out that in the case of War, the limiting probability is $\frac{1}{e^{3/2}} \approx 0.22313$. Further details are available in "Avoiding Your Spouse at a Party Leads to War," published in Mathematics Magazine, a journal of the Mathematical Association of America [**1**].

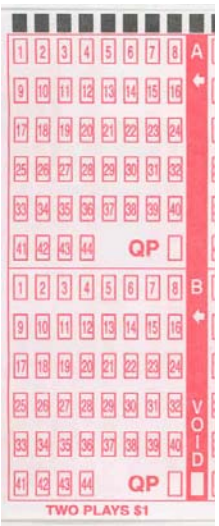
The second issue, that of a never-ending game, was particularly suited to undergraduate level research, and was given to two of my best students. Specifically, they studied the question of whether or not a game can go into an "infinite loop," so that neither player would ever end up holding all the cards. The questions were sufficiently open-ended to allow the research to involve the sort of investigation where examples were studied, conjectures were made, and *then* theorems were proved. A fair amount of programming in Mathematica was done, so that the computer could simulate thousands of games of cards, determining whether a particular deal of the cards would result in a game that ends up in a loop or not. Subsequently, specific games that ended in loops were chosen and the computer printed out the entirety of the games. We were thus able to study the structure of the loops and to see patterns that otherwise might never have been found. The result: under certain conventions of play, it turns out that not only are loops possible, but their structure can be described quite elegantly, as my students did in two individual honors theses.

Gambling. An introductory course in probability and statistics, such as MAT105, is not the first place one might expect to run into a conundrum of the sort that leads to an article, but it happens. Various state lotteries can be used to illustrate the basic concepts of (sophisticated) counting and probability. They are "real world" examples, and the students get some satisfaction from seeing that the answers we arrive at in class match those printed on the play slips.

To a mathematician, there is no difference in the computations from one lotto game to another (only the numbers change; not the concept), so I see no need to work out the answers in advance—I know I'll get them right in class. Such was the case when I asked the students to compute the probability of matching 5 of 6 numbers in the Missouri Lotto. The exact answer is

$$\Pr(\text{match 5 of 6}) = \frac{\binom{6}{5}\binom{38}{1}}{\binom{44}{6}} = \frac{57}{1764763}.$$

The students found that answer, and by taking the reciprocal of $\frac{57}{1764763}$, found that $\Pr(\text{match 5 of 6}) \approx \frac{1}{30960.75}$. They correctly claimed that the odds of matching 5 of 6 numbers were, to the nearest integer, 30961:1. This is the number that *should* have been on the play slip. They then turned over their play slips only to discover that 30961:1 was nowhere to be found. (So much for the students' sense of satisfaction.) After a brief moment of confusion, a more careful look at the play slips revealed that in the Missouri Lotto, a player must buy *two* tickets at \$.50 each, and the odds are given *per \$1 play*:



HOW TO WIN

Match-(Per Play Board)	Win	Odds-(Per $1 Play)
6 of 6	Share of jackpot	1:3,529,526
5 of 6	Share of 2nd prize (avg. $750)	1:15,480
4 of 6	Share of 3rd Prize (avg. $30)	1:335
3 of 6	Free Lotto ticket	1:21

Prizes are 50% of sales. Overall odds 1:20. All Lotto prizes except for free Lotto ticket are pari-mutuel.

Since two games were played, the students suggested that

$$\Pr(\text{match 5 of 6}) = 2 \cdot \frac{57}{1764763} = \frac{114}{1764763} \approx \frac{2}{30960.75} \approx \frac{1}{15480.38}.$$

The play slip stated the odds were 15480:1, and the students were satisfied. At this point, class was over and everyone went home happy. Everyone but me, that is. In general, it is not true that $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$. If it were, then purchasing enough tickets (say 30961 of them) would *guarantee* a win. In order to find the correct probability for the Missouri Lotto problem, it is necessary to consider the fact that both tickets could match 5 of the 6 numbers. The correct answer is given by

$$\Pr(A \cup B) = 2 \cdot \frac{57}{1764763} - \left(\frac{\binom{6}{5}\binom{38}{1}}{\binom{44}{6}} \right)^2 = \frac{201179733}{3114388446169} \approx \frac{1}{15480.\mathbf{62}},$$

an answer which we arrived at next class, after a discussion of the probability rules. Of course the practical difference between these numbers is negligible—if you expect to wait 15480 weeks before winning such a prize, what's another week? On the other hand, in the interest of honest advertising, the odds should be stated on the play slip as 15481:1.

This discrepancy seemed worthy of further investigation. The result: "A Tale of Two Tickets," published as a Classroom Capsule in The College Mathematics Journal (another publication of the MAA). [**2**] In that paper, it was shown that the discrepancy can never be larger than that found in the Missouri Lotto. Formally, the following result was proved.

> **Theorem:** Suppose we have a game in which a player choosing k of n numbers purchases two tickets, selecting the numbers at random. Let A be the event "first ticket matches j of n numbers," and B be the event "second ticket matches j of n numbers." Let the approximate probability $P(A \cup B) \approx P(A) + P(B)$ be expressed as s:1, and the correct probability $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ be expressed as t:1, where s and t are rounded to the nearest integer. Then $t - s = 0$ or $t - s = 1$.

It can also be shown that no matter how many tickets might hypothetically be required, the approximate odds and the correct odds will never differ by more than 1.

Drugs. No life is so complicated that adding a dog to the mix can't bring new problems. And so it was with Rhapsody, the first, and most neurotic, of the five dogs I've lived with. Among Rhapsody's quirks in her later years was a debilitating fear of thunderstorms. During such a storm she could typically be found in the bathtub "digging" for all she was worth, looking for a place to hide. Occasionally, she would crawl into a kitchen cabinet and cower behind the pots and pans[1]:

[1] I *am* capable of focusing a camera, but not when stumbling out of bed without my glasses to deal with a pathetically panting dog in a cabinet at two in the morning.

During these episodes, Rhapsody was inconsolable. As the frequency of such events increased, something had to be done. Rhapsody's vet recommended a daily 20mg dose of Clomicalm, or as I liked to call it, "puppy Paxil." All that was left to do was buy the pills, I naively thought. But then I went on line, searched for Clomicalm and found the following ad [**4**]:

Clomicalm For Dogs 20mg 30 Tablets **Best Price: $22.23** **Buy It!**
Clomicalm For Dogs 40mg 30 Tablets **Best Price: $24.56** **Buy It!**

Having some familiarity with the concept of unit pricing, I chose to purchase the 40mg pills. But now I had to break the pills, which gave me problems. Specifically,

a) What is the expected number of days until we select a half-pill from the bottle?
b) What is the probability that on day 59 there is a whole pill left?
c) What is the expected number of days until the last pill is broken?

One practical way to solve these problems is to come home from the vet, dump the pills on the counter and break them all. This solution is mathematically uninteresting. Instead, we assume that each day Rhapsody takes half a pill and that on any given day a selection is made at random from the pill bottle. If the selection is a half-pill, it is given to Rhapsody; if the selection is a full pill, the pill is cut in half. One half is given to Rhapsody, and the other is put back in the bottle.

There are two reasonable models that can be applied.

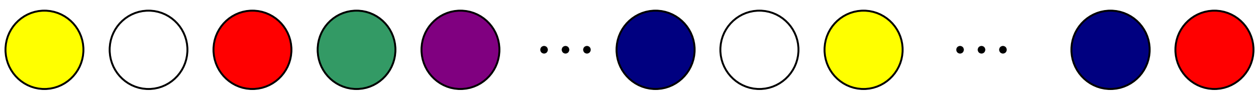
1. Assume that the probability of selecting any particular whole pill is twice that of selecting any particular half pill.
2. Assume that each pill and half pill is equally likely to be selected.

Only the first model will be discussed here. Furthermore, it will be assumed that instead of a specific number of pills (30), there is an arbitrary number, n. That way the limiting behavior can be discussed, as in The Hat Check Problem and War. To aid with understanding of the problem and its solutions, we reformulate it in the traditional setting of marbles and urns.

Imagine an urn containing two marbles of each of n different colors. The two marbles of a given color represent the two halves of a given pill. Marbles are selected one at a time at random without replacement. The three questions can now be rephrased thus:

a) How long can one expect to wait to remove a second marble of some color?
b) What is the probability that when only two marbles are left, they are the same color?
c) How long can one expect to wait until the urn no longer contains two marbles of the same color?

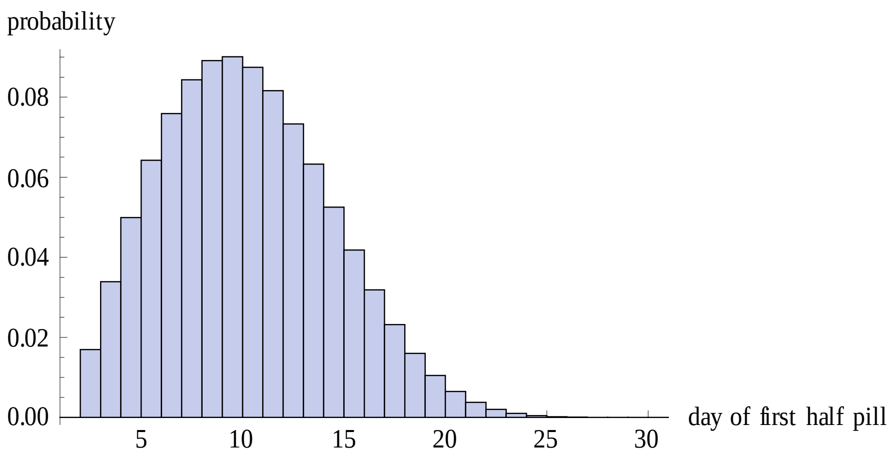
Let X represent the draw on which the first repeated color occurs. To address question a), consider the orderings of the $2n$ marbles in which the first repeated color is at position k.



↑

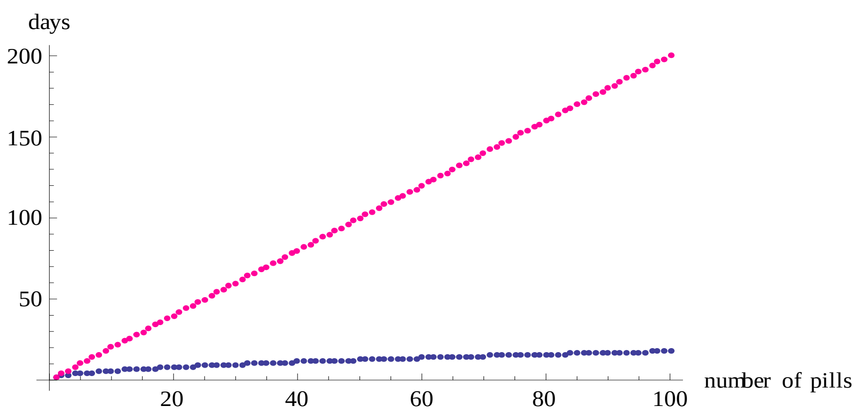
k

The probability that the first time we remove a second marble of a particular color is on the k^{th} try is $P(X = k) = \dfrac{2^{k-1}\binom{n}{k-1}(k-1)(k-1)!(2n-k)!}{(2n)!}$. For Rhapsody's pills, n = 30, the probabilities are shown in the histogram below. In a probability histogram, the height of a bar gives the probability of a particular outcome. In this example, the left-most bar represents finding a half-pill on day two. That event has a probability of about 0.017 (a 1.7% chance of occurring).

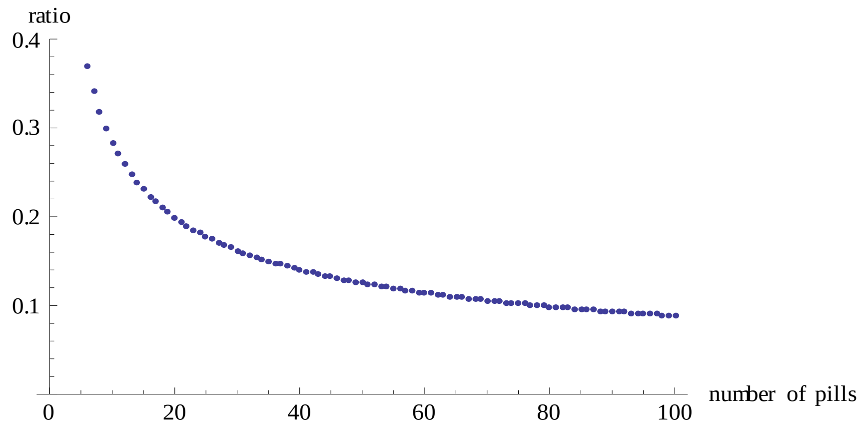


Note that it is impossible to select a second marble of a given color on the first try, as well as on any try after the 31st (One cannot select more than 30 different colored marbles, so a match must occur on or before the 31st draw.)

The expected number of marbles removed before drawing the second of some color is given by $E(X) = \dfrac{2^{2n}\, n!\, n!}{(2n)!}$. When n = 30, the expected value is approximately 9.74866. The interpretation of this number is that, while it will vary pill bottle to pill bottle, over many years the average number of days I will have to wait after starting a new bottle, until that lucky day when I don't have to break a pill in the morning, is 9.74866. Below is a graph of $E(X)$ vs. the number of pills (in blue) along with the numbers $2n$ (in red).

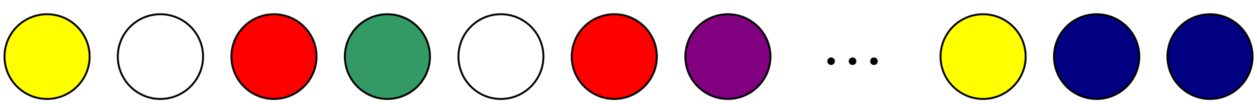


It appears that the number of days one expects to wait to find a half-pill grows very slowly. However, $\lim_{n\to\infty}\frac{2^{2n}n!n!}{(2n)!}=\infty$, which means that if we could buy bottles containing more and more pills, there would be no bound to how long we would have to wait for that half-pill. On the other hand, it is true that the expected number of days to wait grows very slowly compared to the number of days the pills will last. The following graph shows the ratio of $E(X)$ to $2n$.



In fact, $\lim_{n\to\infty}\frac{E(X)}{2n}=0$, so that the proportion of the time a pill bottle will last until we get a half-pill gets arbitrarily small.

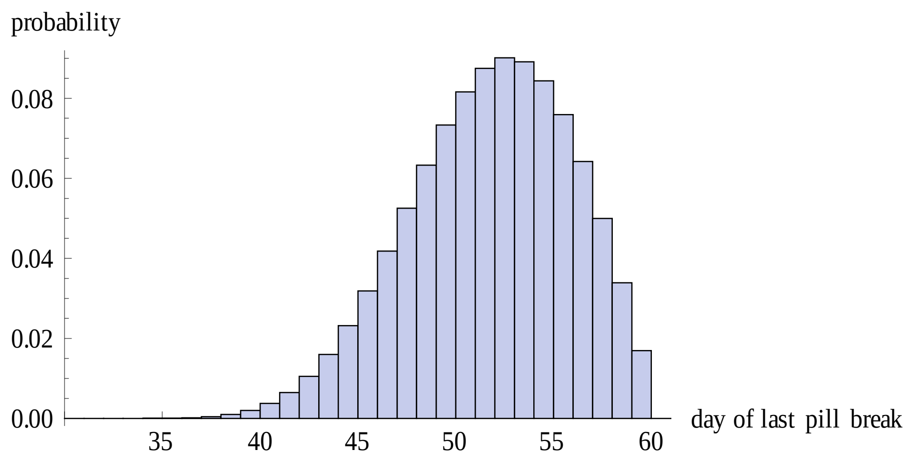
Now consider question b), what is the probability that when only two marbles are left, they are the same color? In this case, orderings of the marbles such as



in which the last two marbles are the same color (representing two halves of the same pill) are relevant. The probability of such an ordering is $=\frac{1}{2n-1}$. For n = 30, the chance of having a whole pill left at 59 is 1/59 or approximately 0.0169492.

More generally, one can ask for the probability that the last pill is broken on day k. Let Y represent the draw on which the last pill is broken. Then

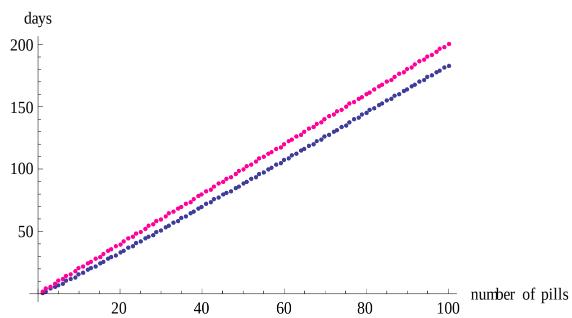
$$P(Y=k)=\frac{2^{2n-k}\binom{n-1}{k-n}(k-1)!(2n-k)!}{(2n)!}.$$ The probabilities for n = 30 pills are shown in the probability histogram below. Since it is impossible to break the last pill before day 30, or on day 60, the probabilities for these values of k are zero.

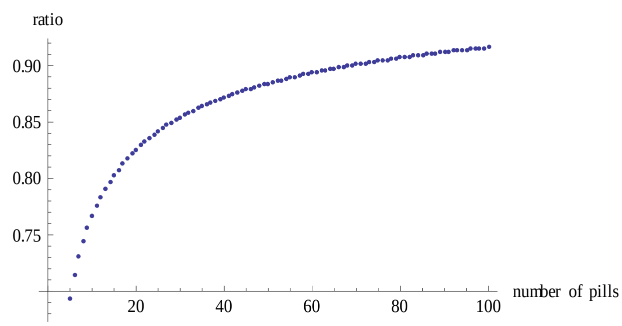


The expected number of days until the last pill is broken is then given by the expression

$$E(Y) = \sum_{k=n}^{2n} \frac{k \cdot 2^{2n-k} \binom{n-1}{k-n} (k-1)!(2n-k)!}{(2n)!}.$$

When n = 30, the value is $E(Y)$ = 51.2513, which as one might expect is rather late in the 60 day process. The behavior for general n is shown in the graph on the left below (in blue) along with the numbers $2n$ (in red). On the right is the ratio $E(Y)/2n$.



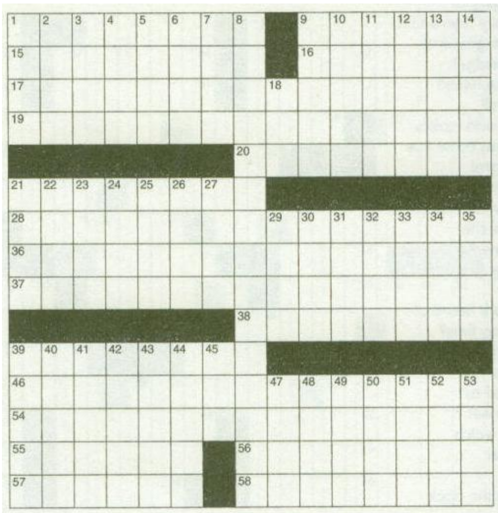


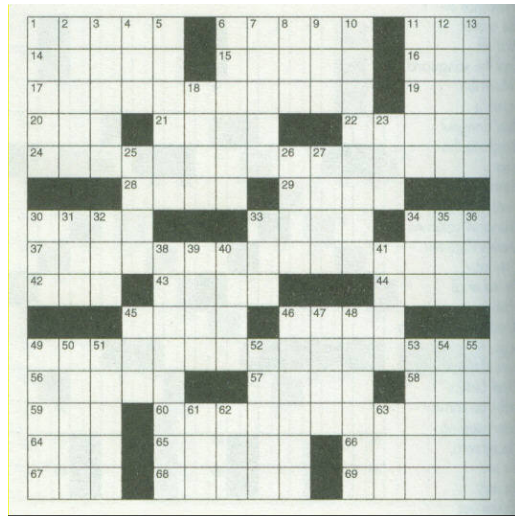
The expression for $E(Y)$ does not simplify nicely as did the expression for $E(X)$, but all numerical evidence at this point indicates that $\lim_{n\to\infty} \frac{E(Y)}{2n} = 1$. It appears that for bottles with a large number of pills, the proportion of the time we must wait until all pills are broken approaches 100% as a limit. This work has been presented to the Sectional Meeting of the Mathematical Association of America, but many questions remain unanswered. In fact, in the process of my putting together the graphs, I noticed that the two histograms may be exact mirror images of each other. Something new to investigate!

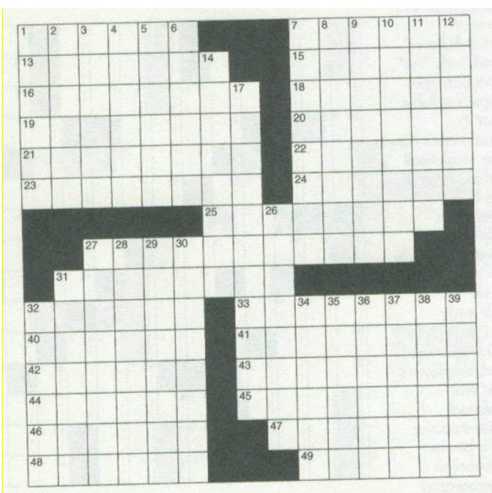
Crosswords. My current obsession is crossword puzzles. I have spent so much time looking at the grids that I recently began to wonder just how many different grids are possible. A standard New York Times crossword puzzle is a 15×15 array of black and white squares, so that there are 225 total squares. Since each square can be one of two colors, the total number of possibilities is given by

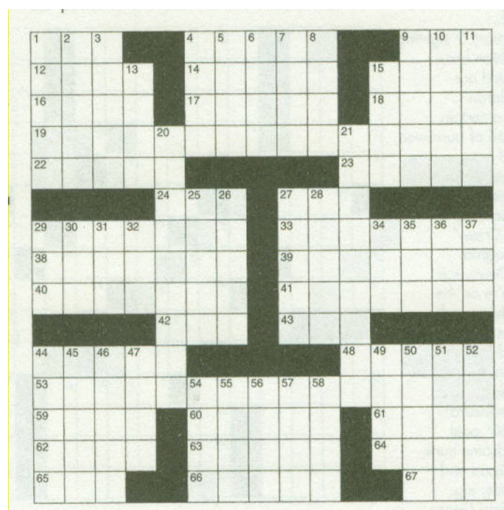
$2^{225} = 53919893334301279589334030174039261347274288845081144962207220498432$.

However, not all arrangements of black and white squares are acceptable as crosswords. Most solvers would be shocked to open up the paper and find only two white squares, for example. Thus we must decide exactly which of the myriad possible arrays are acceptable. In other words, we must define exactly what a crossword puzzle is, mathematically. A quick glance at some puzzles will help.





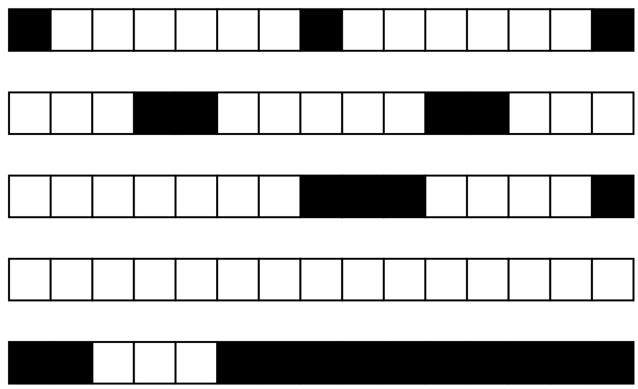




These puzzles all have certain features in common, motivating the following as the mathematical definition of a crossword puzzle:

- An $n \times n$ array of black and white squares
- The white squares are connected
- Each row (and column) has at least one run of white squares
- Every run of white squares has length at least three
- Rotational Symmetry

As far as I can tell at this stage in my work, determining the number of crossword puzzles is going to be extremely difficult, and a long-term project. As is often the case in mathematical endeavors, a modest start was necessary, and I began with the determination of the number of possible rows. In keeping with our definition above, each row must contain at least one run of white squares, and every run of white squares must have length at least three. A few such rows are shown.



One approach to counting the number of rows is recurrence—find a rule that will give the number of rows of length 15, for example, if the number of rows of each shorter length is known. It turns out that such a procedure is successful in this problem. To get started, let

a_n : number of acceptable rows of length n

b_n : number of acceptable rows of length n starting with a black square

w_n : number of acceptable rows of length n starting with a white square

and note that $a_n = b_n + w_n$.

If a row starts with a black square, then that square can be removed, resulting in an acceptable row of length $n - 1$; conversely to any acceptable row of length $n - 1$, a black square can be added. Thus $b_n = a_{n-1}$, so that

$$a_n = a_{n-1} + w_n \qquad (1)$$

Similar reasoning leads to

$$w_n = a_{n-4} + w_{n-1} + 1 \qquad (2)$$

Combining equations (1) and (2) gives the desired recurrence relation:

$$a_n = 2a_{n-1} - a_{n-2} + a_{n-4} + 1. \qquad (3)$$

If the number of rows of length $n - 1$, $n - 2$, and $n - 4$ are known, the number of rows of length n can be computed. It is a simple matter to list all rows of length 3, 4, 5 and 6, so the number of rows of any desired length can be found. Starting with $n = 3$, the sequence

1, 3, 6, 10, 16, 26, 43, 71, 116, 188, 304, 492, 797, …

gives the number of possible rows in an $n \times n$ crossword.

Upon finding a sequence such as that above, it is advisable to check the On-Line Encyclopedia of Integer Sequences [**5**] to determine whether the sequence has been found by someone else working on the same problem, or perhaps as the solution to a different problem. In this case, the sequence was unknown, so I submitted it. This sequence is now officially sequence A130578, further evidence that this is a good problem.

A recurrence relation is nice, but if one wants to know how many rows of length 100 are possible, the number of rows of every length up to 100 would have to be computed. It would be preferable to have an explicit formula. There are standard techniques for solving recurrence relations such as (3), which in this case yield the impressive

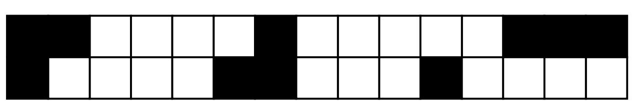
$$a_n = \frac{1}{60\left(-2+\sqrt{5}\right)}\left(3\left(-20\left(-2+\sqrt{5}\right)-\left(-5+\sqrt{5}\right)\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n+\left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^n\left(-25+11\sqrt{5}\right)\right)+30\left(-2+\sqrt{5}\right)\mathrm{Cos}\left[\frac{n\pi}{3}\right]-10\sqrt{3}\left(-2+\sqrt{5}\right)\mathrm{Sin}\left[\frac{n\pi}{3}\right]\right)$$

as the formula for the number of possible rows of length n. Thus the number of rows of length 100 is (in slightly simplified form)

$$\frac{30-30\sqrt{5}-30\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^{100}+12\sqrt{5}\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^{100}+15\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^{100}+3\sqrt{5}\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^{100}}{30\left(-1+\sqrt{5}\right)}$$

or 463686346096539499587.

The natural next step is to determine how many two-row arrangements, such as the one shown below, are possible.



In this case the recurrence relation is a bit more complicated: $\tau_n = 2\tau_{n-1} - \tau_{n-2} + \tau_{n-3} + \tau_{n-4} + \varphi_n$, where $\varphi_n = a_n^2 - 2a_{n-1}^2 + a_{n-2}^2 - a_{n-3}^2 - a_{n-4}^2 - 2a_{n-3}$ and a_n is the number of possible rows as given above. It is a fairly simple matter to implement this sequence on a computer to find the sequence (starting with $n = 3$),

1, 9, 36, 98, 246, 646, 1777, 4883, 13120, 34642, 90976, 239160, 629427 …

OEIS: A133226. On the other hand, an explicit formula is at this point unknown. Mathematica ran for 24 hours trying to find one with no luck.

It seems certain that a recurrence relation for three-row arrangements can be found, but due to the complexity of the two-row case, it may not be the best path to pursue. I have programmed Mathematica to count such arrangements by brute force, but this is extremely inefficient, and in fact the program's memory capacity is exceeded even at the relatively small value $n = 12$. If I want to know about the standard 15×15 puzzle, I'm going to have to find a new approach.

Thankfully, after all this time, I've still got problems.

References:

[1] Brodie, Marc. "Avoiding Your Spouse at a Party Leads to War" *Mathematics Magazine* June, 2002

[2] Brodie, Marc. "A Tale of Two Tickets" *The College Mathematics Journal* May, 2004.

[3] Brualdi, Richard. *Introductory Combinatorics*, 3rd edition, Prentice Hall, New Jersey, 1999.

[4] http://store.yahoo.com/cgi-bin/clink?entirelypets+gVWDFG+clomicalm1.html

[5] *The On-Line Encyclopedia of Integer Sequences* http://www.research.att.com/~njas/sequences/